

**UNIT 1**  
**CALCULUS - I****Module-1 : Calculus**

- Rolle's theorem.
- Mean Value theorems.
- Expansion of functions by Mc.Laurin's for one variable.
- Expansion of functions by Taylor's for one variable.
- Taylor's theorem for function of two variables.
- Partial Differentiation.
- Maxima & Minima (two and three variables).
- Method of Lagranges Multipliers.

**ROLLE'S THEOREM:-**

1. Statement. If  $f(x)$  is a function of the variable  $x$  such that

**(i)**  $f(x)$  is continuous in the closed interval  $[a, b]$ ,

**(ii)**  $f(x)$  is differentiable for every point in the open interval  $(a, b)$  i.e.,  $f'(x)$  exists for every point in the open interval  $(a, b)$  and

**(iii)**  $f(a) = f(b)$  [or as a special case  $f(a) = f(b) = 0$ ] then there is at least one point  $x = c$  in the open interval  $(a, b)$  such that

$$f'(c) = 0.$$

**Proof. Analytical method.**

Condition (i) shows that the function  $f(x)$  does not break at any point in the closed interval  $[a, b]$ .

Condition (iii) i.e.,  $f(a) = f(b)$  shows that either  $f(x)$  is constant in the interval  $[a, b]$ , or  $f(x)$  should either increase or decrease as  $x$  takes values from  $a$  to  $b$ .

In case (a), when  $f(x) = \text{constant}$  then  $f'(x) = 0 \forall x \in (a, b)$ . Hence the theorem is proved in this case.

In case (b), suppose  $f(x)$  increases. But  $f(a) = f(b)$ , therefore,  $f(x)$  must cease to increase and begin to decrease at some point  $x = c$  in  $(a, b)$ , consequently  $f(x)$  will be maximum at  $x = c$ . Hence

$$f(c + h) - f(c) \text{ and } f(c - h) - f(c).$$

are both negative, where  $h$  is small and  $h > 0$ . Hence

$$\frac{f(c + h) - f(c)}{h} < 0 \text{ or } \frac{f(c - h) - f(c)}{-h} > 0.$$

Taking limit as  $h \rightarrow 0$  of both, we have

$$Rf'(c) < 0 \text{ and } Lf'(c) > 0.$$

These both limits are not equal, therefore,  $f'(x)$  does not exist at  $x = c$ , but it is contrary to the condition (ii), so we must have  $Rf'(c) = 0$  and  $Lf'(c) = 0$ . Hence we have

$$f'(c) = 0 \text{ where } a < c < b.$$

### Geometrical Proof:

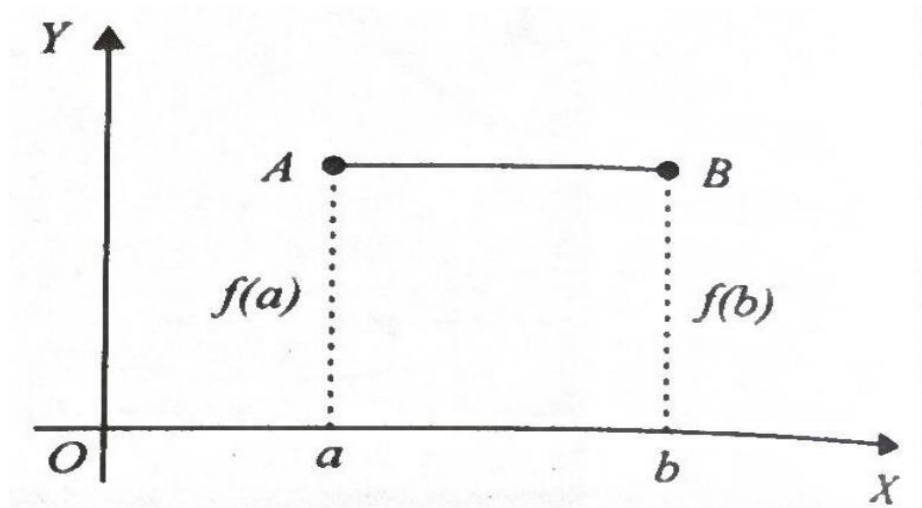
Geometrically the theorem is almost clear. According to condition (i), the function is continuous in the closed interval  $[a, b]$ , therefore, the function can be represented by continuous curve in the interval. Following two cases are possible :

Case I. If  $f(x)$  is constant then

$$f(a) = f(b) = f(x).$$

In this case the function  $f(x)$  represents a straight line parallel to  $x$ -axis.

Hence  $f'(x) = 0 \forall x \in (a, b)$ .

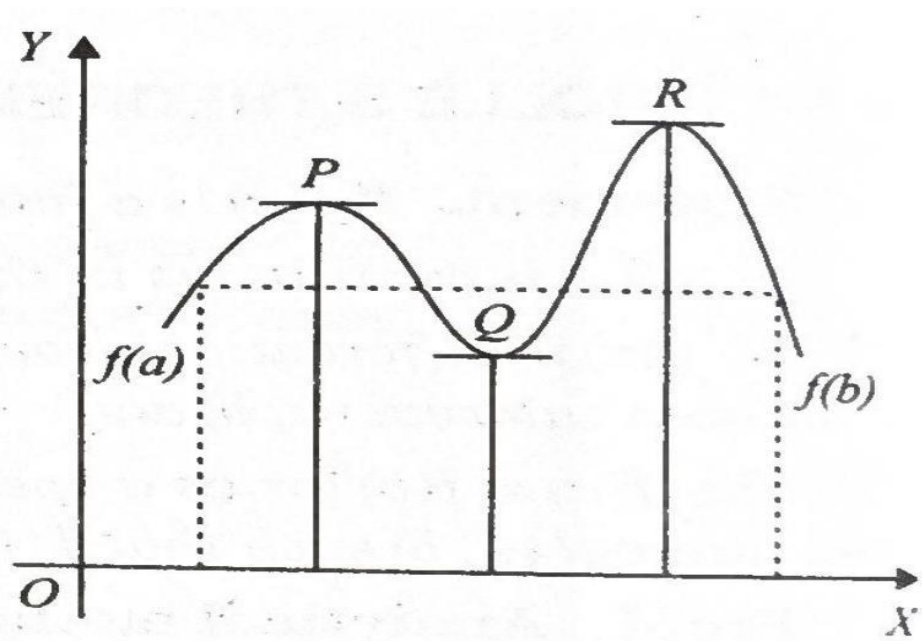


Case II. If the function  $f(x)$  is not continuous. When  $x$  is assigned values from  $a$  to  $b$  then  $f(x)$  will first increase and then decrease or it will first decrease and then increase since

$$f(a) = f(b).$$

Hence there definitely exists at least one point  $c \in (a, b)$  at which  $f(x)$  will cease to increase and then begin to decrease. At this point  $x = c$  [See point  $P, Q$  or  $R$  in the figure] the tangent to the curve  $y = f(x)$  will be parallel to  $x$ -axis. Hence

$$f'(c) = 0 \text{ where } c \in (a, b).$$



Example. State Rolle's theorem and explain its geometrical meaning.

**ILLUSTRATIVE QUESTIONS**

Q 1. (a) Verify Rolle's theorem, where  $f(x) = 2x^3 + x^2 - 4x - 2$ .

**Solution.** Here  $f(x)$  is a rational integral function of  $x$ , therefore  $f(x)$  is continuous and differentiable for all real values of  $x$ . Hence conditions (i) and (ii) of Rolle's theorem are satisfied in any interval.

Now

$$\begin{aligned} f(x) &= 2x^3 + x^2 - 4x - 2 \\ &= x^2(2x + 1) - 2(2x + 1) = (x^2 - 2)(2x + 1) \\ f(x) = 0 &\Rightarrow (x^2 - 2)(2x + 1) = 0 \\ &\Rightarrow x = -\frac{1}{2}, \sqrt{2}, -\sqrt{2} \\ \therefore f\left(-\frac{1}{2}\right) &= f(\sqrt{2}) = f(-\sqrt{2}) = 0 \end{aligned}$$

$f'(x)$  exists for all above values of  $x$ . Now we take the interval  $[-\sqrt{2}, \sqrt{2}]$  so that all conditions of Rolle's theorem are satisfied in this interval. We have now to prove that there exist at least one point ( $x = c$ ) in the open interval  $(-\sqrt{2}, \sqrt{2})$  at which  $f'(x) = 0$ .

$$\text{Now } f'(x) = 6x^2 + 2x - 4 = 0 \Rightarrow 3x^2 + x - 2 = 0$$

$$\Rightarrow (3x - 2)(x + 1) = 0 \Rightarrow x = -1, \frac{2}{3}$$

$$\therefore -\sqrt{2} < -1 < \sqrt{2} \text{ and } -\sqrt{2} < 2/3 < \sqrt{2}.$$

$$\therefore c = -1, 2/3.$$

Hence Rolle's theorem is verified.

Q 1. (b) Verify Rolle's theorem for the function  $f(x) = x^3 - 12x$  in the interval  $0 \leq x \leq 2\sqrt{3}$ .

**Solution.** Here  $f(x) = x^3 - 12x$

$$\begin{aligned} f(x) = 0 &\Rightarrow x^3 - 12x = 0 \Rightarrow x(x^2 - 12) = 0 \\ &\Rightarrow x = 0, \pm 2\sqrt{3} \end{aligned}$$

Now  $f(x)$  is a rational integral function of  $x$ , therefore  $f(x)$  is continuous and differentiable for all real values of  $x$ . So  $f(x)$  is

- (i) Continuous in the closed interval  $[0, 2\sqrt{3}]$ ,  
and (ii) Differentiable in the open interval  $]0, 2\sqrt{3}[$ .

$$\text{Also } f(0) = f(2\sqrt{3}) = 0.$$

Thus all conditions of Rolle's theorem are satisfied. Consequently there exists at least one point ( $x = c$ ) in the open interval  $]0, 2\sqrt{3}[$  at which  $f'(x) = 0$ .

$$\text{Now } f'(x) = 3x^2 - 12 = 0 \Rightarrow x = \pm 2.$$

$$\therefore 0 < 2 < 2\sqrt{3}, \quad \therefore c = 2.$$

Hence Rolle's theorem is satisfied.

**Q 2. Verify Rolle's theorem for  $f(x) = x^3 - 4x$ .**

**Solution.** Here  $f(x) = x^3 - 4x = x(x - 2)(x + 2)$

$$f(x) = 0 \Rightarrow x = 0, 2, -2.$$

$$\therefore f(0) = f(2) = f(-2) = 0.$$

All conditions of Rolle's theorem are satisfied (as in example 1 above). Now we take the intervals  $[-2, 0]$  and  $[0, 2]$  so that there must exist values of  $x$  in the open intervals  $(-2, 0)$  and  $(0, 2)$  such that  $f'(x) = 0$ .

$$\text{Now } f'(x) = 3x^2 - 4 = 0 \Rightarrow x = \pm 2/\sqrt{3}$$

$$\therefore -2 < \frac{2}{\sqrt{3}} < 0 \text{ and } 0 < \frac{2}{\sqrt{3}} < 2$$

$$\therefore c = \frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$$

Hence Rolle's theorem is verified.

Q 3. (a) Discuss the applicability of Rolle's theorem on the function

$$f(x) = \begin{cases} x^2 + 1, & \text{when } 0 \leq x \leq 1 \\ 3 - x, & \text{when } 1 < x \leq 2 \end{cases}$$

**Solution.** (i) To test continuity of  $f(x)$  at  $x = 1$

$$\begin{aligned} f(1-0) &= \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} (1-h)^2 + 1 = 2 \\ f(1+0) &= \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} 3 - (1+h) = 2 \\ f(1) &= 1^2 + 1 = 2 \end{aligned}$$

Thus

$$f(1-0) = f(1+0) = f(1)$$

$\therefore f(x)$  is continuous at  $x = 1$  and therefore  $f(x)$  is continuous in  $[0,2]$ . So first condition of Rolle's theorem is satisfied.

(ii) To test differentiability of  $f(x)$  at  $x = 1$

$$\begin{aligned} Lf'(1) &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{(1-h)^2 + 1 - 2}{-h} = \lim_{h \rightarrow 0} 2 - h = 2 \\ Rf'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{3 - (1+h) - 2}{h} = -1 \\ Lf'(1) &\neq Rf'(1) \end{aligned}$$

$\therefore f'(x)$  does not exist at  $x = 1$  and therefore  $f(x)$  is not differentiable for all  $x \in ]0,2[$ . So second condition of Rolle's theorem is not satisfied.

Hence Rolle's theorem can not be applied on  $f(x)$ .

Example 3. (b) Discuss the conditions of Rolle's theorem for the function  $y = \tan x$  in  $0 \leq x \leq \pi$ .

$$\begin{aligned} \text{Solution. Here } f(x) &= \tan x \\ f(x) = 0 &\Rightarrow \tan x = 0 \Rightarrow x = 0, \pi \\ \therefore f(0) &= f(\pi) \end{aligned}$$

But the function is not continuous at  $x = \frac{1}{2}\pi$  in  $(0, \pi)$  and consequently the function is not differentiable at this point.

Hence first two conditions of Rolle's theorem are not satisfied.

**Example 4. Verify Rolle's theorem for the function**

$$f(x) = x^3 - 6x^2 + 11x - 6$$

**Solution.** Here  $f(x)$  is a rational integral function of  $x$ , therefore  $f(x)$  is continuous and differentiable for all real values of  $x$ . Hence conditions (i) and (ii) of Rolle's theorem are satisfied in any interval.

Now

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 11x - 6 \\ &= x^2(x - 1) - 5x(x - 1) + 6(x - 1) \\ &= (x - 1)(x^2 - 5x + 6) \\ &= (x - 1)(x - 2)(x - 3) \\ f(x) = 0 &\Rightarrow (x - 1)(x - 2)(x - 3) = 0 \\ &\Rightarrow x = 1, 2, 3 \\ f(1) = f(2) = f(3) &= 0 \end{aligned}$$

$f'(x)$  exists for all above values of  $x$ . Now we take the interval  $[1, 3]$  so that all conditions of Rolle's theorem are satisfied in this interval.

Now, we have to prove that there exists at least one point ( $x = c$ ) in the open interval  $(1, 3)$  at which  $f'(x) = 0$ .

Now

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 11 = 0 \\ \Rightarrow x &= \frac{12 \pm \sqrt{144 - 4 \times 3 \times 11}}{2 \times 3} = 2 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

Since

$$1 < \left(2 - \frac{1}{\sqrt{3}}\right) < 2 \text{ and } 1 < \left(2 + \frac{1}{\sqrt{3}}\right) < 3.$$

$\therefore$

$$c = 2 - \frac{1}{\sqrt{3}}, 2 + \frac{1}{\sqrt{3}}$$

Hence Rolle's theorem is verified.

**Example 5. (a) Verify the Rolle's theorem in the interval  $[-1,1]$  for the function  $f(x) = x^2$ .**

**Solution.** Here  $f(x) = x^2$ .

$$\therefore f(-1) = f(1) = 1$$

and

$$f'(x) = 2x$$

Clearly  $f'(x)$  exists for all values of  $x$  in the open interval  $(-1,1)$ . Also  $f(x)$  is continuous for all values of  $x$  in the closed interval  $[-1,1]$ .

Now  $f'(x) = 2x = 0 \Rightarrow x = 0$ .

$\therefore -1 < 0 < 1, \therefore f'(c) = 0$  for  $c = 0$ .

Hence Rolle's theorem is verified.

**Q 5. (b) Can Rolle's theorem be applied for the function  $f(x) = 1 - (x - 3)^{2/3}$ .**

**Solution.** Here

$$\begin{aligned} f(x) &= 1 - (x - 3)^{2/3} \\ f(x) = 0 &\Rightarrow 1 - (x - 3)^{2/3} = 0 \end{aligned}$$

Now

$$(x - 3)^{2/3} = 1 \Rightarrow (x - 3)^2 = (1)^3$$

$$\Rightarrow (x - 3)^2 = 1 \Rightarrow x - 3 = \pm 1 \Rightarrow x = 2, 4.$$

$$\therefore f(2) = f(4) = 0.$$

Now

$$f'(x) = -\frac{2}{3}(x - 3)^{-1/3} = \frac{-2}{3(x - 3)^{1/3}}$$

Clearly  $f'(x)$  does not exist at  $x = 3 \in (2,4)$ . Thus conditions of Rolle's are not satisfied in the interval  $(2, 4)$ . Hence Rolle's theorem cannot be applied to the given function.

**Example 6.** Verify Rolle's theorem for the function  $f(x) = x(x + 3)e^{-(1/2)x}$  in  $[-3,0]$ .  
**Solution.**

$$f(x) = x(x + 3)e^{-(1/2)x} = (x^2 + 3x)e^{-(1/2)x}$$

$$f(-3) = f(0) = 0$$

$\therefore f(x)$  is differentiable in the open interval  $(-3,0)$ . Also  $f(x)$  is continuous for all values of  $x$  in  $[-3,0]$ .

Hence all conditions of Rolle's theorem are satisfied.

Let  $x = c$ ,  $-3 < c < 0$  such that  $f'(c) = 0$ .

Now

$$f'(x) = (2x + 3)e^{-(1/2)x} + (x^2 + 3x)e^{-(1/2)x} \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}(-x^2 + x + 6)e^{-(1/2)x}.$$

$$\therefore f'(x) = 0 \Rightarrow \frac{1}{2}(-x^2 + x + 6)e^{-(1/2)x} = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x - 3)(x + 2) = 0$$

$$\Rightarrow x = 3, -2.$$

Clearly  $-2 \in (-3,0)$  and  $3 \notin (-3,0)$ .

$\therefore c = -2$ .

Hence Rolle's theorem is verified.

### **LAGRANGE'S MEAN VALUE THEOREM OR FIRST MEAN VALUE THEOREM:-**

**Statement.** If  $f(x)$  is a function of the variable  $x$  and ( i )  $f(x)$  is continuous in the closed interval  $[a, b]$ , (ii)  $f(x)$  is differentiable in the open interval  $( a, b )$  i.e.,  $f'(x)$  exists in the open interval  $( a, b )$ . then there exists at least one point  $c$  in the open interval  $( a, b )$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

**Proof.** Consider a function  $F(x)$  defined as follows:

$$F(x) = f(x) + A \cdot x \tag{1}$$

where  $A$  is a constant to be determined such that

$$\begin{aligned} F(a) &= F(b) \\ \therefore f(a) + A \cdot a &= f(b) + A \cdot b \end{aligned}$$

or

$$-A = \frac{f(b) - f(a)}{b - a} \quad (2)$$

Now since  $f(x)$  is continuous in the closed interval  $[a, b]$  and  $x$  is continuous in every interval. Hence  $F(x)$  is also continuous in  $[a, b]$ . Again  $f(x)$  is differentiable in the open interval  $(a, b)$  and  $x$  is differentiable in every interval. Hence  $F(x)$  is also differentiable in  $(a, b)$ . Therefore,  $F(x)$  satisfy all conditions of Rolle's theorem in the interval  $(a, b)$ . Consequently, there exists at least one point  $c$  in the open interval  $(a, b)$  such that

$$\begin{aligned} F'(c) &= 0 \\ \text{From (1), } F'(x) &= f'(x) + A \\ \therefore F'(c) &= f'(c) + A \end{aligned} \quad (4)$$

From (3) and (4), we have

$$0 = f'(c) + A \Rightarrow -A = f'(c). \quad (5)$$

Hence from (2) and (5), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (6)$$

Note. If  $b - a = h$  and  $0 < \theta < 1$  then  $a + \theta h$  is a point in the interval  $(a, a + h)$ .

The result (6) above, can be represented by

or

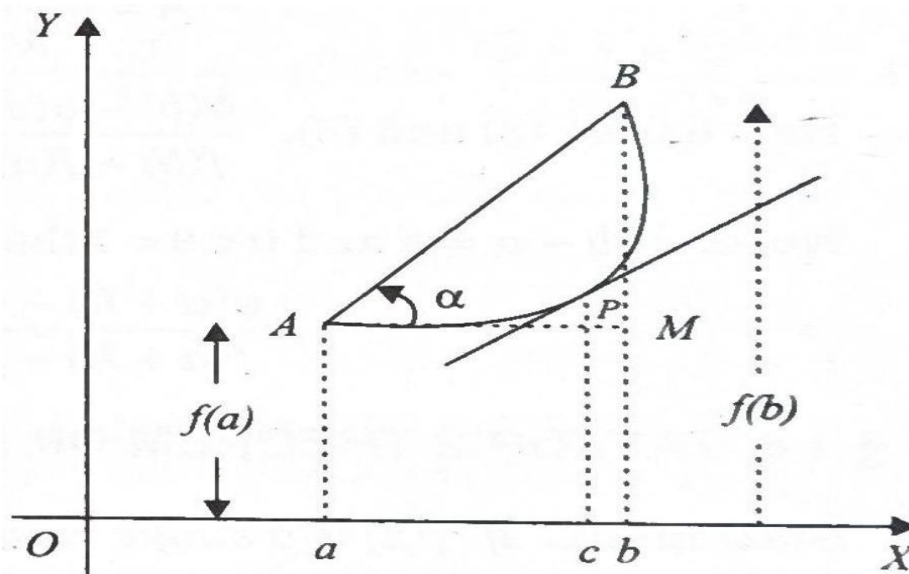
$$\begin{aligned} f'(a + \theta h) &= \frac{f(a + h) - f(a)}{h} \\ f(a + h) - f(a) &= hf'(a + \theta h) \end{aligned}$$

### Geometrical Interpretation :

Suppose that the graph of the curve  $y = f(x)$  is represented by  $APB$  in the figure. Let the chord  $A$  make on angle  $\alpha$  with  $x$ -axis, where

$$\begin{aligned}\tan \alpha &= \frac{BM}{AM} \\ \Rightarrow \tan \alpha &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow \tan \alpha &= f'(c) \text{ where } c \in (a, b)\end{aligned}$$

[by the mean value theorem]



It shows that tangent to the curve at the point  $x = c$  is parallel to the chord  $AB$  where

$$a < c < b.$$

### ILLUSTRATIVE QUESTIONS

**Q 1.** Verify mean value theorem for The function  $x^3$  in  $[-2,2]$ ,

**Solution.** Let  $f(x) = x^3, \therefore f'(x) = 3x^2$

Now the mean value theorem will be verified if there exist a value '  $c$  ' of  $x$  in  $[-2,2]$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ where } a = -2, b = 2$$

i.e.,

$$\frac{(2)^3 - (-2)^3}{2 - (-2)} = 3c^2$$

$$\text{Or } \frac{8 - (-8)}{4} = 3c^2 \text{ or } 4 = 3c^2$$

$$\text{Or } c^2 = 4/3 \text{ or } c = \pm 2/\sqrt{3}$$

Clearly  $c = \pm 2/\sqrt{3}$  lies in  $[-2, 2]$ .

Hence Lagrange's mean value theorem is verified.

**Q 2. Verify Lagrange's mean value theorem for the function  $f(x) = (x - 3)(x - 6)(x - 9)$  on the interval  $[3, 5]$ .**

**Solution.** Here  $f(x) = (x - 3)(x - 6)(x - 9)$ , and  $a = 3, b = 5$ .

(i) Since  $f(x)$  has a definite and unique value  $\forall x \in [3, 5]$ , therefore,  $f(x)$  is continuous in  $[3, 5]$ .

(ii) Clearly

$$f(x) = x^3 - 18x^2 + 99x - 162$$

$$\therefore f'(x) = 3x^2 - 36x + 99$$

Clearly  $f'(x)$  exists  $\forall x \in ]3, 5[$ .

Thus all conditions of Lagrange's mean value theorem are satisfied, and so  $c \in ]3, 5[$  [such that

$$\begin{aligned} \frac{f(5) - f(3)}{5 - 3} &= f'(c) \\ \Rightarrow \frac{8 - 0}{2} &= 3c^2 - 36c + 99 \\ \Rightarrow 3c^2 - 36c + 95 &= 0 \\ \Rightarrow c &= \frac{36 \pm \sqrt{(36)^2 - 4 \times 3 \times 95}}{2 \times 3} = 6 \pm \frac{\sqrt{39}}{3} \\ &= 6 \pm \frac{\sqrt{6.24}}{3} = 6 \pm 2.08 = 3.92, 8.08 \end{aligned}$$

$$\text{But } 3.92 \in ]3, 5[ (3) = 0]$$

Hence Lagrange's mean value theorem is verified.

**Q 3. Verify Lagrange's mean value theorem for the function  $f(x) = \sqrt{x^2 - 4}$  in the interval  $[2, 4]$ .**

**Solution.** Here  $f(x) = \sqrt{x^2 - 4}$ , and  $a = 2, b = 4$ .

(i) Since  $f(x)$  has a definite and unique value  $\forall x \in [2, 4]$ , therefore,  $f(x)$  is continuous in  $[2, 4]$ .

(ii)  $f'(x) = \frac{1}{2}(x^2 - 4)^{-1/2} \cdot (2x) = \frac{x}{\sqrt{x^2 - 4}}$ . Clearly  $f'(x)$  exists  $\forall x \in ]2, 4[$ .

Thus all conditions of Lagrange's mean value theorem are satisfied, and so  $c \in ]2, 4[$  such that

$$\begin{aligned} \frac{f(4) - f(2)}{4 - 2} &= f'(c) \\ \Rightarrow \frac{\sqrt{12} - 0}{2} &= \frac{c}{\sqrt{c^2 - 4}} \\ \Rightarrow 4c^2 &= 12(c^2 - 4) \Rightarrow c = \pm\sqrt{6} \end{aligned}$$

But

$$\sqrt{6} \in ]2, 4[.$$

Hence Lagrange's mean value theorem is verified.

### **CAUCHY'S MEAN VALUE THEOREM OR SECOND MEAN VALUE THEOREM**

**Statement.** If two functions  $f(x)$  and  $\phi(x)$  are

- (i) Continuous in the closed interval  $[a, b]$ ,
- (ii) Differentiable in the open interval  $(a, b)$ ,
- (iii)  $f'(x) \neq 0$  at any point in the open interval  $(a, b)$  then there exists at least one point  $x = c$  in the open interval  $(a, b)$  such that

$$\frac{\phi(b) - \phi(a)}{f(b) - f(a)} = \frac{\phi'(c)}{f'(c)}, \text{ where } a < c < b$$

**Q. Verify Cauchy's mean value theorem for the functions  $x^2$  and  $x^3$  in  $[1, 2]$ .**

**Solution.** Let  $f(x) = x^2$  and  $g(x) = x^3$ .

The function  $f(x)$  and  $g(x)$  both are continuous and differentiable in the closed interval  $[1,2]$  and  $g'(x) = 3x^2 \neq 0$  for every value of  $x$  in the interval  $(1,2)$ .

We have now to verify that there exist at least one value '  $c$  ' of  $x$  in the open interval  $(1,2)$  such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \text{ where } a = 1, b = 2 \quad (1)$$

Now,

$$f(a) = f(1) = (1)^2 = 1 \text{ and } f(b) = f(2) = (2)^2 = 4$$

$$f'(x) = 2x, \therefore f'(c) = 2c \text{ and } g'(x) = 3x^2, \therefore g'(c) = 3c^2.$$

Putting values in (1), we get

$$\frac{4 - 1}{8 - 1} = \frac{2c}{3c^2} \text{ or } \frac{3}{7} = \frac{2}{3c}$$

$$\therefore 9c = 14 \text{ or } c = 14/9$$

Clearly  $c = 14/9$  lies in  $(1,2)$ .

Hence Cauchy's mean value theorem is verified.

### Maclaurin's Theorem:-

Statement. If  $f(x)$  is a function of  $x$  such that represented as an infinite power series centered at  $x=0$ , where each term is a derivative of the function evaluated at zero, multiplied by a power of  $x$  divided by the corresponding factorial is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x) \quad (5)$$

where  $0 < \theta < 1$ .

It is called Maclaurin's Theorem.

### ILLUSTRATIVE QUESTIONS

Q 1. Prove that  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$ .

**Solution.** By Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^n(\theta x)$$

Now,

$$\begin{aligned} f(x) &= e^x, \therefore f(0) = e^0 = 1 \\ f'(x) &= e^x, \therefore f'(0) = e^0 = 1 \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ f^{(n-1)}(x) &= e^x, \therefore f^{(n-1)}(0) = e^0 = 1 \\ f^n(x) &= e^x, \therefore f^n(\theta x) = e^{\theta x}. \end{aligned}$$

Substituting these values in Maclaurin's expansion

$$\begin{aligned} e^x &= 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 1 + \dots + \frac{x^{n-1}}{(n-1)!} \cdot 1 + \frac{x^n}{n!} e^{\theta x} \\ e^x &= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x} \end{aligned}$$

Q 2. Prove that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n-2}x^{n-1}}{(n-1)} + \frac{(-1)^{n-1}x^n}{n(1+\theta x)^n}$$

Solution. Let

$$\begin{aligned} f(x) &= \log(1+x) \\ f^n(x) &= \frac{(-1)^{n-1} \cdot (n-1)!}{(1+x)^n} (1) \end{aligned}$$

Putting  $x = 0$  in (1),  $f^n(0) = (-1)^{n-1} \cdot (n-1)!$

Put  $n = 1, 2, 3, \dots, (n-1)$  in (2), we get

$$\begin{aligned} f'(0) &= (-1)^0 \cdot 0! = 1 \\ f''(0) &= (-1)^1 \cdot 1! = -1 \\ f'''(0) &= (-1)^2 \cdot 2! = 2! \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ f^{(n-1)}(0) &= (-1)^{n-2} \cdot (n-2)! \end{aligned}$$



and

$$f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$\begin{aligned} \log(1+x) &= \log(1+0) + x \cdot 1 + \frac{x^2}{2!} \cdot (-1) + \frac{x^3}{3!} \cdot 2! + \dots \\ &+ \frac{x^{n-1}}{(n-1)!} (-1)^{n-2} \cdot (n-2)! + \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n-2}x^{n-1}}{(n-1)} + \frac{(-1)^{n-1}x^n}{n(1+\theta x)^n} \end{aligned}$$

**Q 3. Apply Maclaurin's theorem to prove that**

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots$$

**Solution.** Let  $f(x) = \log \sec x$ ,

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= 1 \\ f'''(0) &= 0 \\ f^{(iv)}(0) &= 2 \\ f^{(v)}(0) &= 0 \\ f^{(vi)}(0) &= 16. \end{aligned}$$

By Maclaurin's theorem, we have

$$\begin{aligned} \log \sec x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \frac{x^5}{5!}f^{(v)}(0) + \frac{x^6}{6!}f^{(vi)}(0) + \dots \\ &= 0 + x \cdot 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot 2 + \frac{x^5}{5!} \cdot 0 + \frac{x^6}{6!} \cdot 16 + \dots \\ &= \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots \end{aligned}$$

Q 4. Expand  $e^{a\sin^{-1}x}$  by Maclaurin's theorem and find the general term.

Hence show that  $e^x = 1 + \sin \theta + \frac{1}{2!} \sin^2 \theta + \frac{2}{3!} \sin^3 \theta + \dots$

**Solution.** Let  $y = e^{a\sin^{-1}x}$ ,

$$(y)_0 = 1$$

$$\begin{aligned} \therefore y_1 &= e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} \\ \Rightarrow (1-x^2)y_1^2 &= a^2y^2, \end{aligned}$$

$$(y_1)_0 = a.$$

Differentiating again, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = a^2 \cdot 2yy_1$$

i.e.,

$$(1-x^2)y_2 - xy_1 - a^2y = 0, (y_2)_0 = a^2$$

Differentiating  $n$  times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Putting

$$x = 0, (y_n + 2)_0 = (n^2 + a^2)(y_n)_0. \quad (1)$$

Now substituting  $n = 1, 2, 3, 4, 5 \dots$  in equation (1), we have

$$\begin{aligned} (y_3)_0 &= a(1^2 + a^2) \\ (y_4)_0 &= a^2(2^2 + a^2) \\ (y_5)_0 &= a(1^2 + a^2)(3^2 + a^2) \end{aligned}$$

$\therefore$  By Maclaurin's theorem, we have

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1^2 + a^2)x^3}{3!} + \dots \quad (2)$$

Putting  $a = 1, x = \sin \theta$  in (2), we get

$$e^\theta = 1 + \sin \theta + \frac{\sin^2 \theta}{2!} + \frac{2\sin^3 \theta}{3!} + \dots$$

**Q 5. Find the first five terms in the expansion of  $e^{\sin x}$  by Maclaurin's theorem.**

**Solution.** Let,  $y = e^{\sin x}$ ,

$$(y)_0 = 1$$

Differentiating it successively, we get :

$$y_1 = \cos x \cdot e^{\sin x}$$

i.e.,

$y_1 = y \cos x,$	$(y_1)_0 = 1$
$y_2 = y_1 \cos x - y \sin x,$	$(y_2)_0 = 1$
$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$ $= y_2 \cos x - 2y_1 \sin x - y \cos x,$	$(y_3)_0 = 0$
$y_4 = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x - y_1 \cos x + y \sin x$ $= y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$	$(y_4)_0 = -3$
$y_5 = y_4 \cos x - 4y_3 \sin x - 6y_2 \cos x + 4y_1 \sin x + y \cos x$	$(y_5)_0 = -8$

∴ By Maclaurin's theorem,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \dots$$

**Q 6. Expand  $\tan^{-1} x$  in ascending powers of  $x$  by Maclaurin's theorem.**

Solution. Let $y = \tan^{-1} x,$	$(y)_0 = \tan^{-1} 0 = 0$
∴ $y_1 = \frac{1}{1+x^2},$	$(y_1)_0 = \frac{1}{1+0} = 1$
$(1+x^2)y_1 = 0$	
∴ $(1+x^2)y_2 + 2xy_1 = 0,$	$(y_2)_0 = 0.$

Differentiating (1)  $n$  times by Leibnitz's theorem, we get

$$(1+x^2)y_{n+1} + {}^n C_1 y_n (2x) + {}^n C_2 y_{n-1} (2.1) = 0$$

or

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$$

Putting  $x = 0,$

$$(y_{n+1})_0 = -(n - 1)n(y_{n-1})_0. \tag{2}$$

Putting  $n = 2, 3, 4, \dots$  in (2),

$$\begin{aligned} (y_3)_0 &= -1 \cdot y_1(y_1)_0 = -1 \cdot 2(1) = -2! \\ (y_4)_0 &= -2 \cdot 3(y_2)_0 = 0 & (1) \\ (y_8)_0 &= -3 \cdot 4(y_3)_0 = 3 \cdot 4 \cdot 2! = 4! & (1) \\ (y_8)_0 &= 0(y_7)_0 = -8 \cdot 0(y_0)_0 = -5 \cdot 0 \cdot 4! = 0, 0 \neq 0. (1) \end{aligned}$$

∴ By Maclaurin's theorem, we get

$$\begin{aligned} \tan^{-1} x &= 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-2!) + 0 + \frac{x^5}{5!}(4!) + 0 + \frac{x^7}{7!}(-3!) + \dots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)} + \dots \end{aligned}$$

**Q 7. Expand by Maclaurin's theorem  $\frac{e^x}{1+e^x}$  as far as the term  $x^3$ .**

**Solution.** Here  $y = \frac{e^x}{1+e^x} = \frac{1+e^x-1}{1+e^x} = 1 - \frac{1}{1+e^x}$ .

$$\begin{aligned} \therefore (y)_0 &= \frac{e^0}{1+e^0} = \frac{1}{2} \\ y_1 &= 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x} \cdot \frac{1}{1+e^x} = y(1-y) = y - y^2 \\ (y_1)_0 &= \frac{1}{4} \\ y_2 &= y_1 - 2yy_1 \\ \therefore (y_2)_0 &= \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = 0 \\ \therefore y_3 &= y_2 - 2yy_2 - 2y_1^2 \\ (y_3)_0 &= -1/8 \text{ etc.} \end{aligned}$$

∴ By Maclaurin's theorem, we have

$$\frac{e^x}{1+e^x} = \frac{1}{2} + x \cdot \frac{1}{4} + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \left(-\frac{1}{8}\right) + \dots = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$$

**TAYLOR'S THEOREM:-**

**Statement.** If  $f(x)$  is a function of  $x$  (Where  $x$  is independent of  $h$ ) such that all the derivatives upto  $n$  th, ascending power of  $x$  then

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(h) + \frac{x^n}{n!}f^{(n)}(h), \text{ where } 0 < \theta < 1.$$

**ILLUSTRATIVE QUESTIONS**

**Q 1.** Expand  $\sin x$  in powers of  $\left(x - \frac{1}{2}\pi\right)$ .

**Solution.** Let  $f(x) = \sin x$ , we can write

$$f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$$

By Taylor's theorem, we have

$$\begin{aligned} f(x) &= f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right] \\ &= f\left(\frac{1}{2}\pi\right) + \left(x - \frac{1}{2}\pi\right)f'\left(\frac{1}{2}\pi\right) + \frac{\left(x - \frac{1}{2}\pi\right)^2}{2!}f''\left(\frac{1}{2}\pi\right) + \dots \end{aligned}$$

Now  $f(x)$

$$\begin{aligned} \therefore \sin x &= \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f''\left(\frac{1}{2}\pi\right) \\ \therefore f'(x) & \end{aligned}$$

$$= \sin \frac{1}{2}\pi = 1, f'\left(\frac{1}{2}\pi\right) = 0, f''\left(\frac{1}{2}\pi\right) = -1, f'''\left(\frac{1}{2}\pi\right) = 0, f^{(4)}\left(\frac{1}{2}\pi\right) = 1.$$

Putting these values in (1), we have

$$\begin{aligned} \sin x &= 1 + \left(x - \frac{1}{2}\pi\right) \cdot 0 + \frac{\left(x - \frac{1}{2}\pi\right)^2}{2!} \cdot (-1) + \frac{\left(x - \frac{1}{2}\pi\right)^3}{3!} \cdot 0 + \frac{\left(x - \frac{1}{2}\pi\right)^4}{4!} \cdot 1 + \dots \\ &= 1 - (1/2!) \left(x - \frac{1}{2}\pi\right)^2 - (1/4!) \left(x - \frac{1}{2}\pi\right)^4 \dots \end{aligned}$$

**Q 2. Expand  $\tan\left(x + \frac{\pi}{4}\right)$  as far as the term  $x^4$  and evaluate  $\tan 46.5^\circ$  to four significant digits.**

**Solution.** Let

$$f(x) = \tan x \quad \therefore f\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$$

Now,

$$f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + [f(x)]^2 \quad \therefore f'\left(\frac{\pi}{4}\right) = 1 + 1 = 2$$

$$f''(x) = 2f(x)f'(x) \quad \therefore f''\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 2f(x)f''(x) + 2[f'(x)]^2$$

$$\therefore f'''\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f''\left(\frac{\pi}{4}\right) + 2\left[f'\left(\frac{\pi}{4}\right)\right]^2 = 2 \times 1 \times 4 + 2 \times 4 = 16$$

$$f^{iv}(x) = 2f(x)f'''(x) + 6f'(x)f''(x)$$

$$\therefore f^{iv}\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f'''\left(\frac{\pi}{4}\right) + 6f'\left(\frac{\pi}{4}\right)f''\left(\frac{\pi}{4}\right) = 2 \times 1 \times 16 + 6 \times 2 \times 4 = 80$$

By Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots \tag{1}$$

Putting  $h = x, a = \pi/4$  in (1), we get

$$\begin{aligned} f\left(\frac{\pi}{4} + x\right) &= f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!}f^{iv}\left(\frac{\pi}{4}\right) + \dots \\ \Rightarrow \tan\left(x + \frac{\pi}{4}\right) &= 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \frac{x^4}{4!}(80) + \dots \\ \Rightarrow \tan\left(x + \frac{\pi}{4}\right) &= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots \tag{2} \end{aligned}$$

$$\Rightarrow \tan(x + 45^\circ) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$$

[Note.  $\tan 45^\circ = \tan(\pi/4) = f(\pi/4) =$

Now putting  $x = 1.5^\circ = \frac{3}{2} \times \frac{\pi}{180}$  radians = 0.02618 in (3), we get

$$\begin{aligned} \tan(1 \cdot 5^\circ + 45^\circ) &= 1 + 2(0 \cdot 02618) + 2(0 \cdot 02618)^2 + \frac{8}{3}(0 \cdot 02618)^3 + \dots \\ \Rightarrow \tan 46 \cdot 5^\circ &= 1 \cdot 05378 \end{aligned}$$

**Q 3. Expand  $\tan^{-1}x$  in powers of  $(x - \frac{1}{4}\pi)$ .**

**Solution.** Let  $f(x) = \tan^{-1}x$ .

$$\therefore f(x) = f\left[\frac{1}{4}\pi + \left(x - \frac{1}{4}\pi\right)\right].$$

$\therefore$  By Taylor's theorem, we have

$$f(x) = f\left(\frac{1}{4}\pi\right) + \left(x - \frac{1}{4}\pi\right)f'\left(\frac{1}{4}\pi\right) + \frac{\left(x - \frac{1}{4}\pi\right)^2}{2!}f''\left(\frac{1}{4}\pi\right) + \dots \quad (1)$$

Now

$$f(x) = \tan^{-1}x, f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}, f''(x) = -2x/(1+x^2)^2, \text{ etc.}$$

$$\therefore f\left(\frac{1}{4}\pi\right) = \tan^{-1}\frac{1}{4}\pi, f'\left(\frac{1}{4}\pi\right) = 1/(1+1/16\pi^2), f''\left(\frac{1}{4}\pi\right) = -\pi/[2\{1+(1/16)\pi^2\}^2],$$

etc.

Putting these values in (1), the required expansion is given by

$$\tan^{-1}x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x - \frac{1}{4}\pi\right)\frac{1}{(1+\pi^2/16)} - \frac{\left(x - \frac{1}{4}\pi\right)^2}{2!} \cdot \frac{\pi}{2(1+\pi^2/16)^2} + \dots$$

**Q 4. Compute the approximate value of  $\sqrt{11}$  to four decimal places by taking for the five terms of an appropriate Taylor's expansion.**

$$f(x) = \sqrt{x}, \quad \therefore f(x+a) = \sqrt{(x+a)}$$

$$\text{Now, } f'(x) = \frac{1}{2}x^{-1/2},$$

Solution. Let

$$f'(x) = \frac{1}{2} \left(-\frac{1}{2}\right) x^{-3/2} = -\frac{1}{4} x^{-3/2} = -\frac{15}{16} x^{-7/2}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}, \quad f^{iv}(x)$$

By Taylor's series, we have

$$f(x+a) = f(x) + af'(x) + \frac{a^2}{2!} f''(x) + \frac{a^3}{3!} f'''(x) + \dots$$

$$\therefore \sqrt{(x+a)} = \sqrt{x} + a \cdot \frac{1}{2\sqrt{x}} - \frac{a^2}{8} x^{-3/2} + \frac{3a^3}{48} x^{-5/2} - \frac{15a^4}{384} x^{-7/2}$$

Putting  $x = 9, a = 2$ , we have

$$\sqrt{9+2} = \sqrt{9} + \frac{1}{\sqrt{9}} - \frac{1}{2 \times 9\sqrt{9}} + \frac{1}{2 \times 81\sqrt{9}} - \frac{5}{8 \times 729\sqrt{9}} + \dots$$

$$\Rightarrow \sqrt{11} = 3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{486} - \frac{5}{17496} + \dots$$

$$= 3 + 0.33333 - 0.01852 + 0.00206 - 0.00029$$

$$= 3.31658 = 3.3166 \text{ correct to four decimal places}$$

**Q 5. Expand  $e^x$  in powers of  $(x-1)$ .**

**Solution.** Let  $f(x) = e^x$  then

$$f(x) = f[1 + (x-1)]$$

By Taylor's theorem, we have

$$e^x = f(x) = f[1 + (x-1)] = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!} f''(1) + \dots \quad (1)$$

Now

$$f(x) = e^x, f'(x) = e^x, f''(x) = e^x \text{ etc.}$$

Therefore

$$f(1) = e, f'(1) = e, f''(1) = e, \text{ etc.}$$

Putting these values in (1), we have

$$e^x = e + (x - 1)e + \frac{(x - 1)^2}{2!}e + \frac{(x - 1)^3}{3!}e + \dots$$

### **PARTIAL DIFFERENTIATION**

Let  $z$  be a symbol which has a unique definite value for each pair of values of  $x$  and  $y$ , then  $z$  is said to be a function of two independent variables  $x$  and  $y$ . It is usually written as  $z = f(x, y)$ .

A function of two independent variables  $x$  and  $y$  is also written as  $F(x, y)$  or  $\psi(x, y)$  or  $\phi(x, y)$  etc.

A similar definition can be given for functions of several independent variables.

#### **Partial Derivatives of Higher Orders :**

Let  $z = f(x, y)$  be a function of  $x$  and  $y$ . If  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  exist, then they can be further differentiated partially w.r. to  $x$  or/and  $y$ , thus giving the second order derivative. We write

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial^2 z}{\partial x^2} = z_{xx} = f_{xx} = \lim_{h \rightarrow 0} \frac{f_x(x+h, y) - f_x(x, y)}{h} \\ \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial y^2} = z_{yy} = f_{yy} = \lim_{k \rightarrow 0} \frac{f_y(x, y+k) - f_y(x, y)}{k} \\ \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial^2 z}{\partial y \partial x} = f_{yx} = \lim_{k \rightarrow 0} \frac{f_x(x, y+k) - f_x(x, y)}{k} \\ \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial x \partial y} = f_{xy} = \lim_{h \rightarrow 0} \frac{f_y(x+h, y) - f_y(x, y)}{h} \end{aligned}$$

### **ILLUSTRATIVE QUESTIONS**

Q 1. If  $z(x + y) = x^2 + y^2$ , show that  $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$ .

[R.G.T.U. Dec. 2000, June 2011]

**Solution.** Here

$$z = \frac{x^2 + y^2}{x + y}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{(x + y) \cdot 2x - (x^2 + y^2) \cdot 1}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

Now

$$1 - \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 1 - \frac{4xy}{(x + y)^2} = \frac{(x - y)^2}{(x + y)^2}$$

Again

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2 - y^2)}{(x + y)^2} = \frac{2(x - y)}{x + y}$$

$$(1) \text{ and } (2) \Rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Q 2. If  $u = e^{xyz}$ , show that  $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2) e^{xyz}$ .

[R.G.T.U. June 2004]

**Solution.** Here  $u = e^{xyz}$ .

$$\therefore \frac{\partial u}{\partial z} = e^{xyz} \cdot xy.$$

Again

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial z} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y} [e^{xyz} \cdot xy] = x \frac{\partial}{\partial y} [e^{xyz} \cdot y] \\ &= x[e^{xyz} + y \cdot e^{xyz} \cdot xz] = x(1 + xyz)e^{xyz} = (x + x^2 yz)e^{xyz} \end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial^3 u}{\partial x \partial y \partial z} &= \frac{\partial}{\partial x} \left[ \frac{\partial^2 u}{\partial y \partial z} \right] = \frac{\partial}{\partial x} [(x + x^2 yz)e^{xyz}] \\ &= (1 + 2xyz)e^{xyz} + e^{xyz} \cdot yz(x + x^2 yz) \\ &= [1 + 2xyz + xyz + x^2 y^2 z^2]e^{xyz} = [1 + 3xyz + x^2 y^2 z^2]e^{xyz}\end{aligned}$$

**Example 3.** If  $u = \log \frac{x^4 + y^4}{x + y}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$ .

[R.G.T.U. June 2006]

**Solution.** Here

$$u = \log \frac{x^4 + y^4}{x + y}$$

Differentiating both sides partially w.r. to  $x$ , we get

Similarly,

$$\frac{\partial u}{\partial x} = \frac{1}{(x^4 + y^4)/(x + y)} \cdot \frac{4x^3(x + y) - (x^4 + y^4) \cdot 1}{(x + y)^2} = \frac{3x^4 + 4x^3y - y^4}{(x^4 + y^4)(x + y)}$$

$$\begin{aligned}\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= \frac{1}{(x^4 + y^4)(x + y)} [3x^5 + 4x^4y - xy^4 + 3y^5 + 4y^4x - x^4y] \\ &= \frac{3[x^5 + x^4y + y^4x + y^5]}{(x^4 + y^4)(x + y)} = \frac{3(x^4 + y^4)(x + y)}{(x^4 + y^4)(x + y)} = 3\end{aligned}$$

**Q 4.** If  $u = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$ ,  $xy \neq 0$ ; then prove that  $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ .

[R.G.T.U. June 2010]

**Solution.** Here  $u = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$ .

Differentiating both sides partially w. r. to  $y$ , we have

$$\begin{aligned}\frac{\partial u}{\partial y} &= x^2 \cdot \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1} \frac{x}{y} + y^2 \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{x}{y^2} \\ &= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} = x - 2y \tan^{-1} \frac{x}{y}\end{aligned}$$

Now differentiating  $\frac{\partial u}{\partial y}$  partially w.r. to  $x$ , we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = 1 - 2y \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} = 1 - \frac{2y^2}{y^2 + x^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

Q 5. If  $u = f\left(\frac{y}{x}\right)$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$ .

[R.G.T.U. Dec. 2014]

**Solution.** Here  $u = f\left(\frac{y}{x}\right)$ .

$$\therefore \frac{\partial u}{\partial x} = \left\{ f' \left( \frac{y}{x} \right) \right\} \left( -\frac{y}{x^2} \right). \quad (1)$$

$$\therefore x \frac{\partial u}{\partial x} = -\frac{y}{x} f' \left( \frac{y}{x} \right).$$

Again  $\frac{\partial u}{\partial y} = \left\{ f' \left( \frac{y}{x} \right) \right\} \left( \frac{1}{x} \right)$ .

$$\therefore y \frac{\partial u}{\partial y} = \frac{y}{x} f' \left( \frac{y}{x} \right). \quad (2)$$

(1) and (2)  $\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$ .

Q6. If  $u = (x^2 + y^2 + z^2)^{-1/2}$ ;  $x^2 + y^2 + z^2 \neq 0$  then prove that

(a)  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -u$ ,

[R.G.T.U. Feb. 2005, Jan./Feb. 2007]

(b)  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ .

[R.G.T.U. June 2002, Nov./Dec. 2007]

**Solution.** (a) Here  $u = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$ .

Now differentiating both sides partially w.r. to  $x$ , we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2(x^2 + y^2 + z^2)^{3/2}} \cdot 2x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore x \frac{\partial u}{\partial x} = -\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}} \quad (1)$$

Similarly,

$$y \frac{\partial u}{\partial y} = -\frac{y^2}{(x^2 + y^2 + z^2)^{3/2}} \quad (2)$$

and

$$z \frac{\partial u}{\partial z} = -\frac{z^2}{(x^2 + y^2 + z^2)^{3/2}} \quad (3)$$

Adding (1), (2) and (3), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{1}{(x^2 + y^2 + z^2)^{1/2}} = -u$$

(b) Differentiating  $\frac{\partial u}{\partial x}$  partially w.r. to  $x$ , we get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left\{ -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right\} \\ &= -\left\{ \frac{1 \cdot (x^2 + y^2 + z^2)^{3/2} - x \cdot \frac{3}{2} (x^2 + y^2 + z^2)^{1/2} \cdot 2x}{(x^2 + y^2 + z^2)^3} \right\} \\ &= -\frac{(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \{x^2 + y^2 + z^2 - 3x^2\} \\ &= -\frac{1}{(x^2 + y^2 + z^2)^{5/2}} \{y^2 + z^2 - 2x^2\} \quad (4) \end{aligned}$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}} \{z^2 + x^2 - 2y^2\}$$

and

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}} \{x^2 + y^2 - 2z^2\}$$

Adding (4), (5) and (6), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}} \{y^2 + z^2 - 2x^2 + z^2 + x^2 - 2y^2 + x^2 + y^2 - 2z^2\}$$

$$= 0$$

Q 7. If  $u = \log(x^3 + y^3 + z^3 - 3xyz)$ , prove that

(i)  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ .

[R.G.T.U. June 2007]

(ii)  $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$ .

[R.G.T.U. Jan./Feb. 2008]

(iii)  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$ .

**Solution.** (i) Here  $u = \log_e(x^3 + y^3 + z^3 - 3xyz)$ .

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \tag{2}$$

and

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \tag{3}$$

Adding (1), (2) and (3), we get

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{x^3 + y^3 + z^3 - 3xyz} \\ &= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{(x + y + z)(x^2 + y^2 + z^2 - yz - xz - xy)} = \frac{3}{x + y + z} \tag{4} \end{aligned}$$

(ii)  $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{3}{x + y + z}\right)$$

$$= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} = -\frac{9}{(x + y + z)^2} \tag{4}$$

(iii) From Algebra,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$$

where  $\omega$  is cube root of unity satisfying  $1 + \omega + \omega^2 = 0$  and  $\omega^3 = 1$

Thus

$$u = \log_e(x + y + z) + \log_e(x + y\omega + z\omega^2) + \log_e(x + y\omega^2 + z\omega)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{x + y + z} + \frac{1}{x + y\omega + z\omega^2} + \frac{1}{x + y\omega^2 + z\omega}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x + y + z)^2} - \frac{1}{(x + y\omega + z\omega^2)^2} - \frac{1}{(x + y\omega^2 + z\omega)^2} \quad (5)$$

Again  $\frac{\partial u}{\partial y} = \frac{1}{x + y + z} + \frac{\omega}{x + y\omega + z\omega^2} + \frac{\omega^2}{x + y\omega^2 + z\omega}$ .

$$\therefore \frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x + y + z)^2} - \frac{\omega^2}{(x + y\omega + z\omega^2)^2} - \frac{\omega^4}{(x + y\omega^2 + z\omega)^2} \quad (6)$$

and

$$\frac{\partial u}{\partial z} = \frac{1}{x + y + z} + \frac{\omega^2}{x + y\omega + z\omega^2} + \frac{\omega}{x + y\omega^2 + z\omega} \quad (7)$$

$$\therefore \frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x + y + z)^2} - \frac{\omega^4}{(x + y\omega + z\omega^2)^2} - \frac{\omega^2}{(x + y\omega^2 + z\omega)^2}$$

Adding (5), (6) and (7), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x + y + z)^2} - \frac{1 + \omega^2 + \omega^4}{(x + y\omega + z\omega^2)^2} - \frac{1 + \omega^4 + \omega^2}{(x + y\omega^2 + z\omega)^2}$$

But

$$1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0 \quad [\because \omega^4 = \omega^3 \cdot \omega = \omega]$$

$\therefore$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x + y + z)^2}$$

Q 8. If  $u = f(r)$ , where  $r^2 = x^2 + y^2$ , show that  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$ .

[R.G.T.U. Dec. 2001, June 2006]

**Solution.** It is given that

$$r^2 = x^2 + y^2. \quad (1)$$

Differentiating (1) partially w.r. to  $x$ , we get

Similarly,

$$2r \frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \quad (2)$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad (3)$$

Now  $u = f(r)$  gives

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{x}{r} \quad (2)$$

$$\begin{aligned} \therefore \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left[ \frac{f'(r)x}{r} \right] = \frac{r[f'(r) \cdot 1 + xf''(r) \cdot \frac{\partial r}{\partial x}] - xf'(r) \cdot \frac{\partial r}{\partial x}}{r^2} \\ &= \frac{1}{r^2} \left[ rf'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r) \right]. \end{aligned}$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2} \left[ rf'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r) \right].$$

Adding, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{r^2} \left[ 2rf'(r) + (x^2 + y^2)f''(r) - \frac{(x^2 + y^2)}{r} f'(r) \right] \\ &= \frac{1}{r^2} [2rf'(r) + r^2 f''(r) - rf'(r)] = \frac{1}{r} f'(r) + f''(r) \end{aligned}$$

Q 9. If  $u(x, y, z) = \log(\tan x + \tan y + \tan z)$ , prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

[R.G.T.U. June 2013]

**Solution.** Given  $u = \log(\tan x + \tan y + \tan z)$ .

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} &= \frac{\sec^2 x}{\tan x + \tan y + \tan z} \\ \Rightarrow \sin 2x \frac{\partial u}{\partial x} &= \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z} = \frac{2 \sin x \cos x \left( \frac{1}{\cos^2 x} \right)}{\tan x + \tan y + \tan z} \\ &= \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad (1) \end{aligned}$$

Similarly, by symmetry we have  
and

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad (2)$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad (3)$$

Adding (1), (2) and (3), we get

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

## TAYLOR'S SERIES (TAYLOR'S THEOREM) FOR FUNCTION OF TWO VARIABLES

**Statement.** If  $f(x, y)$  possesses finite and continuous partial derivatives upto  $n$ th order in any neighbourhood  $D$  of a point  $(a, b)$  and let  $(a + h, b + k)$  be any point of  $D$ , then

$$f(a + h, b + k) = f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(a, b) + \frac{1}{3!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f(a, b) + \dots$$

Where  $a \leq x \leq a + h, b \leq y \leq b + k$ .

**ILLUSTRATIVE QUESTIONS**

Q 1. Expand  $e^{x+y}$  in powers of  $x$  and  $y$  as far as terms of third degree.

**Or**

Expand  $e^{x+y}$  about  $(0,0)$  as far as the terms of third degree.

**Solution.** Let

$$\begin{aligned} f(x, y) &= e^{x+y} & \therefore f(0,0) &= 1 \\ f_x(x, y) &= e^{x+y} & \therefore f_x(0,0) &= e^0 = 1 \\ f_y(x, y) &= e^{x+y} & \therefore f_y(0,0) &= 1 \\ f_{xx} &= f_{yy} = f_{xy} = e^{x+y} \end{aligned}$$

$\therefore$  At  $(0,0), f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = 1$

$$f_{xxxx} = f_{xxy} = f_{xyy} = f_{yyy} = e^{x+y}.$$

At  $(0,0)$ ,

$$f_{xxxx} = f_{xxy} = f_{xyy} = f_{yyy} = 1.$$

By Taylor's series, we have

$$f(x, y) = f(0,0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!} \left(x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2}\right) f(0,0) + \frac{1}{3!} \left(x^3 \frac{\partial^3}{\partial x^3} + 3x^2y \frac{\partial^3}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3}{\partial x \partial y^2} + y^3 \frac{\partial^3}{\partial y^3}\right) f(0,0) + \dots \dots (1)$$

On putting values in (1), we get

$$\begin{aligned} e^{x+y} &= 1 + (x + y) + \frac{1}{2!} (x^2 + 2xy + y^2) + \frac{1}{3!} (x^3 + 3x^2y + 3xy^2 + y^3) + \dots \\ &= 1 + (x + y) + \frac{1}{2!} (x + y)^2 + \frac{1}{3!} (x + y)^3 + \dots \end{aligned}$$

**Q 2. Expand  $\cos(x + y)$  in powers of  $x$  and  $y$ .**

**Solution.** Here  $f(x, y) = \cos(x + y) \therefore f(0,0) = 1$

$$f_x = f_y = -\sin(x + y); f_{xx} = f_{xy} = f_{yy} = -\cos(x + y); f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = \sin(x + y)$$

$$\therefore \text{At}(0,0), f_x = f_y = 0, f_{xx} = f_{xy} = f_{yy} = -1; f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = 0$$

By Taylor's series, we have [putting values in equation (1) of Ex. 1. above.]

$$\begin{aligned} \cos(x + y) &= 1 + 0 + \frac{1}{2!} (x^2 + 2xy + y^2)(-1) + 0 + \dots \\ &= 1 - \frac{(x + y)^2}{2!} + \frac{(x + y)^4}{4!} - \dots \end{aligned}$$

**Q 3. Find the expansion of  $f(x, y) = \cos x \cos y$  in power of  $x, y$  upto fourth order terms.**

**Solution.** Here

$$\begin{aligned} f(x, y) &= \cos x \cos y \\ f_x &= -\sin x \cos y \\ f_y &= -\cos x \sin y \end{aligned}$$

$$\begin{aligned} \therefore f(0,0) &= 1 \\ \therefore f_x(0,0) &= 0 \\ \therefore f_y(0,0) &= 0 \end{aligned}$$

$$\begin{aligned} f_{xx} &= -\cos x \cos y \\ f_{xy} &= \sin x \sin y & \therefore \\ f_{yy} &= -\cos x \cos y & \therefore \\ f_{x^3} &= \sin x \cos y & f_{xy}(0,0) = -1 \\ f_{x^2y} &= \cos x \sin y & \therefore \\ f_{xy^2} &= \sin x \cos y & \therefore \\ f_{y^3} &= \cos x \sin y & \therefore f_{x^3}(0,0) = 0 \\ f_{x^4} &= \cos x \cos y & \therefore f_{xy^2}(0,0) = 0 \\ f_{x^3y} &= -\sin x \sin y & \therefore f_{y^3}(0,0) = 0 \\ f_{x^2y^2} &= \cos x \cos y & \therefore f_{x^4}(0,0) = 1 \\ & & \therefore f_{x^3y}(0,0) = 0 \\ f_{y^4} &= \cos x \cos y & \therefore f_{x^2y^2}(0,0) = 1 \\ \text{have} & & \therefore f_{xy^3}(0,0) = 0 \\ & & \therefore f_{y^4}(0,0) = 1 \end{aligned}$$

By Taylor's series, we have

$$f(x, y) = f(0,0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!} \left(x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2}\right) f(0,0) + \frac{1}{3!} \left(x^3 \frac{\partial^3}{\partial x^3} + 3x^2y \frac{\partial^3}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3}{\partial x \partial y^2} + y^3 \frac{\partial^3}{\partial y^3}\right) f(0,0) + \frac{1}{4!} \left(x^4 \frac{\partial^4}{\partial x^4} + 4x^3y \frac{\partial^4}{\partial x^3 \partial y} + 6x^2y^2 \frac{\partial^4}{\partial x^2 \partial y^2} + 4xy^3 \frac{\partial^4}{\partial x \partial y^3} + y^4 \frac{\partial^4}{\partial y^4}\right) f(0,0) + \dots (1)$$

Putting above values in (1), we get

$$\begin{aligned} \cos x \cos y &= 1 + (0 + 0) + \frac{1}{2!} (-x^2 + 0 - y^2) + \frac{1}{3!} (0 + 0 + 0 + 0) \\ &\quad + \frac{1}{4!} (x^4 + 0 + 6x^2y^2 + 0 + y^4) + \dots \\ &= 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \dots \end{aligned}$$

## MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES

### SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA (LAGRANGE'S CONDITION FOR TWO INDEPENDENT VARIABLES)

Theorem. Let  $\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0$  and  $\left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$

Let

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} = r, \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} = s \text{ and } \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} = t. \text{ Then}$$

- (i) if  $(rt - s^2) > 0$  and  $r > 0$ ,  $f(x, y)$  is minimum at  $(a, b)$ ;
- (ii) if  $(rt - s^2) > 0$  and  $r < 0$ ,  $f(x, y)$  is maximum at  $(a, b)$ ;
- (iii) if  $(rt - s^2) < 0$ ,  $f(x, y)$  is neither maximum nor minimum at  $(a, b)$ ;
- (iv) if  $(rt - s^2) = 0$  the case is doubtful.

### WORKING RULE FOR MAXIMA AND MINIMA OF TWO VARIABLES

Let  $f(x, y)$  be a given function

1. Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

2. Solve the equations  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$  for  $x$  and  $y$ . Then pairs of values of  $x$  and  $y$ , thus obtained will give stationary values of  $f(x, y)$ . Let  $(a, b)$  be one of these pairs.
3. Find  $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$ ;  $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$  and  $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$  and calculate  $rt - s^2$ .
4. (i) if  $(rt - s^2)$  is positive and  $r > 0$ , then  $f(x, y)$  has a minimum at  $(a, b)$ ;  
 (ii) if  $rt - s^2 > 0$  and  $r < 0$  then  $f(x, y)$  has a maximum at  $(a, b)$ ;  
 (iii) if  $(rt - s^2)$  is negative then  $f(x, y)$  has neither maximum nor minimum at  $(a, b)$ . Thus  $(a, b)$  is a saddle point.  
 (iv) if  $rt - s^2 = 0$ , then the case is doubtful and further investigation is required.

### ILLUSTRATIVE QUESTIONS

**Q 1. Discuss the maximum or minimum values of the function**

$$f(x, y) = x^3 - 4xy + 2y^2$$

**Solution.** Here  $\frac{\partial f}{\partial x} = 3x^2 - 4y$  and  $\frac{\partial f}{\partial y} = -4x + 4y$ . For maxima or minima of  $f$ , we have

$$\begin{aligned} \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 &\Rightarrow 3x^2 - 4y = 0, -4x + 4y = 0 \\ \Rightarrow 3x^2 - 4y = 0, x = y &\Rightarrow 3y^2 - 4y = 0, x = y \\ \Rightarrow y(3y - 4) = 0, x = y &\Rightarrow y = 0 \text{ or } y = \frac{4}{3}, x = y \\ \Rightarrow (0,0) \text{ and } \left(\frac{4}{3}, \frac{4}{3}\right) &\text{ are critical points.} \end{aligned}$$

$$\text{Now } r = \frac{\partial^2 f}{\partial x^2} = 6x, s = \frac{\partial^2 f}{\partial x \partial y} = -4, t = \frac{\partial^2 f}{\partial y^2} = 4$$

Thus, at  $(0,0)$  :

$$r = 0, rt - s^2 = (0)(4) - (-4)^2 = -16 < 0.$$

Hence we conclude that  $(0,0)$  is a saddle point of  $f$ . Again at  $\left(\frac{4}{3}, \frac{4}{3}\right)$  :

Since

$$\begin{aligned} r = 8, s = -4, t = 4 \\ r = 8 > 0 \text{ and } rt - s^2 = 8(4) - 16 = 16 > 0 \end{aligned}$$

Hence function  $f$  has a minimum value at  $x = y = 4/3$ . Thus, at  $x = y = 4/3$ .

The minimum value of  $f = f\left(\frac{4}{3}, \frac{4}{3}\right) = -\frac{32}{27}$ .

**Q 2. Discuss the maximum or minimum values of  $u$  in the following cases :**

(i)  $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ .

(ii)  $u = x^3 - y^2 - 3x$ .

**Solution.** (i) Here  $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ .

$$\therefore \frac{\partial u}{\partial x} = y - \frac{a^3}{x^2} \text{ and } \frac{\partial u}{\partial y} = x - \frac{a^3}{y^2}$$

Then for maxima or minima of  $u$ , we have

$$\begin{aligned} \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 &\Rightarrow y - \frac{a^3}{x^2} = 0, x - \frac{a^3}{y^2} = 0 \\ \Rightarrow a^3 = x^2y, a^3 = xy^2 &\Rightarrow a^3 = x^2y = xy^2 \Rightarrow x = y = a. \\ \text{Now } r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3}, s = \frac{\partial^2 u}{\partial x \partial y} = 1 &\text{ and } t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}. \end{aligned}$$

Thus at  $x = y = a$ ; we get

$$\begin{aligned} r = 2, s = 1 \text{ and } t = 2 \\ \therefore rt - s^2 = 2 \cdot 2 - 1^2 = 3 > 0. \end{aligned}$$

Since  $r$  and  $rt - s^2$  are both positive,  $u$  is minimum when  $x = y = a$ .

Hence the minimum value of the function is

$$a^2 + a^2 + a^2 = 3a^2.$$

(ii) Here  $u = x^3 - y^2 - 3x$ . Hence

$$\frac{\partial u}{\partial x} = 3x^2 - 3 \text{ and } \frac{\partial u}{\partial y} = -2y.$$

For maxima or minima of  $u$ , we have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \Rightarrow 3x^2 - 3 = 0, -2y = 0 \Rightarrow x = \pm 1, y = 0$$

Hence the critical points are  $(1,0)$  and  $(-1,0)$ .

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 6x, s = \frac{\partial^2 u}{\partial x \partial y} = 0 \text{ and } t = \frac{\partial^2 u}{\partial y^2} = -2$$

Thus at  $(-1, 0)$ , we get

$$r = -6, s = 0 \text{ and } t = -2$$

$$\therefore rt - s^2 = (-6) \cdot (-1) - 0 = 12 > 0$$

Since  $r < 0$  and  $rt - s^2 > 0$ ,  $u$  is maximum at  $(-1, 0)$ . Hence the maximum value of the function

$$= -1 - 0 + 3 = 2.$$

Now at the point  $(1, 0)$ , we get

If gives

$$r = 6(1) = 6, s = 0 \text{ and } t = -2$$

$$rt - s^2 = 6(-2) - 0 = -12 < 0$$

Therefore  $u$  is neither maximum nor minimum at  $(1, 0)$ . It is a saddle point.

**Q 3. Discuss the maxima and minima of the function  $ax^3y^2 - x^4y^2 - x^3y^3$ .**

**Solution.** Let  $u = ax^3y^2 - x^4y^2 - x^3y^3$ .

$$\therefore \frac{\partial u}{\partial x} = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3 \text{ and } \frac{\partial u}{\partial y} = 2ax^3y - 2x^4y - 3x^3y^2$$

For maxima or minima of  $u$ , we have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \Rightarrow 3a - 4x - 3y = 0$$

$$2a - 2x - 3y = 0 \Rightarrow x = \frac{a}{2}, y = \frac{a}{3}$$

$\therefore$  The point  $\left(\frac{a}{2}, \frac{a}{3}\right)$  is a critical point.

Now at the point  $\left(\frac{a}{2}, \frac{a}{3}\right)$ , we have

$$\begin{aligned}
 r &= \frac{\partial^2 u}{\partial x^2} = 6axy^2 - 12x^2y^2 - 6xy^3 \\
 &= 6xy^2(a - 2x - y) = 6 \cdot \frac{a}{2} \cdot \frac{a^2}{9} \cdot \left(a - a - \frac{a}{3}\right) = -\frac{a^4}{9} \\
 s &= \frac{\partial^2 u}{\partial x \partial y} = 6ax^2y - 8x^3y - 9x^2y^2 \\
 &= x^2y(6a - 8x - 9y) = \frac{a^2}{4} \cdot \frac{a}{3} \cdot (6a - 4a - 3a) = -\frac{a^4}{12} \\
 t &= \frac{\partial^2 u}{\partial y^2} = 2ax^3 - 2x^4 - 6x^3y \\
 &= 2x^3(a - x - 3y) = 2 \cdot \frac{a^3}{8} \cdot \left(a - \frac{a}{2} - a\right) = -\frac{a^4}{8}
 \end{aligned}$$

Now

$$rt - s^2 = \left(\frac{1}{72} - \frac{1}{144}\right)a^8 > 0$$

Since  $r$  is  $-ve$  and  $rt - s^2$  is  $+ve$ , therefore  $u$  in maximum when

$$x = \frac{a}{2}, y = \frac{a}{3}$$

**Q 4. Discuss the extreme values (maxima and minima) of the function  $x^3 + y^3 - 3axy$ .**  
**Solution.** Let

$$u = x^3 + y^3 - 3axy$$

$\therefore$

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial u}{\partial y} = 3y^2 - 3ax$$

Thus for maxima or minima of  $u$

$$\begin{aligned}
 \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 &\Rightarrow 3x^2 - 3ax = 0, 3y^2 - 3ax = 0 \\
 \Rightarrow x^2 = ay, y^2 = ax &\Rightarrow x = y = a.
 \end{aligned}$$

$\therefore$  The point  $(a, a)$  is a critical point.

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 6x, s = \frac{\partial^2 u}{\partial x \partial y} = -3a \text{ and } t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

At the point  $(a, a)$ , we have

If gives

$$r = 6a, s = -3a \text{ and } t = 6a$$

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 27a^2 > 0$$

Since  $rt - s^2$  is +ve and  $r$  is +ve or -ve according as  $a$  is +ve or -ve, we have a maximum or minimum according as  $a$  is -ve or +ve at  $x = y = a$ .

**Q 5. Find all the maximum and minimum values of functions  $f$  given by**

(i)  $f(x, y) = xy(a - x - y).$

(ii)  $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$

**Solution.** (i) For maxima and minima of  $f$ , we have

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 = 0 \tag{1}$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy = 0. \tag{2}$$

Subtraction (2) from (1), we have

$$a(y - x) - (y^2 - x^2) = 0 \Rightarrow (y - x)(a - y - x) = 0 \Rightarrow y = x \text{ or } y + x = a$$

Putting  $y = x$  in (1), we have

$$ya - 2y^2 - y^2 = 0 \Rightarrow 3y^2 - ya = 0 \Rightarrow y = 0, y = a/3$$

$$\text{When } y = a - x, (2) \Rightarrow xa - x^2 - 2x(a - x) = 0 \Rightarrow xa - x^2 - 2ax + 2x^2 = 0$$

$$\Rightarrow x^2 - ax = 0 \Rightarrow x = 0, x = a$$

Thus  $f$  may have maxima or minima at  $(0,0); (a/3, a/3); (0, a); (a, 0).$

Now

$$r = \frac{\partial^2 f}{\partial x^2} = -2y = 0; -\frac{2a}{3}; -2a; 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y = a; -\frac{a}{3}; -a; -a$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x; 0; -\frac{2a}{3}; 0; -2a$$

$$rt - s^2 = -ve; +ve; -ve; -ve.$$

Thus  $f$  has a maximum at  $(a/3, a/3)$  and has no extreme value at other points.

$$\therefore \text{maximum value of } f = \frac{a^2}{9} \left( a - \frac{a}{3} - \frac{a}{3} \right) = \frac{a^3}{27}.$$

(ii) For maxima or minima of  $f$ , we have

$$\frac{\partial f}{\partial x} = 3x^2 - 63 + 12y = 0 \tag{1}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x = 0 \tag{2}$$

Subtracting (2) from (1), we have

$$\begin{aligned} & 3(x^2 - y^2) + 12(y - x) = 0 \\ \Rightarrow & 3(x - y)(x + y) - 12(x - y) = 0 \\ \Rightarrow & 3(x - y)(x + y - 4) = 0 \Rightarrow x = y \text{ or } x + y = 4. \end{aligned}$$

(i) Putting  $x = y$  in (1), we get

$$\begin{aligned} & 3x^2 + 12x - 63 = 0 \Rightarrow x^2 + 4x - 21 = 0 \\ \Rightarrow & (x + 7)(x - 3) = 0 \Rightarrow x = -7 \text{ or } 3 \end{aligned}$$

Thus  $f$  may have maxima or minima at  $(-7, -7)$  and  $(3, 3)$ .

$$\begin{aligned} & 3y^2 - 63 + 12(4 - y) = 0 \Rightarrow 3y^2 - 12y - 15 = 0 \\ \Rightarrow & y^2 - 4y - 5 = 0 \Rightarrow (y - 5)(y + 1) = 0 \quad [\because x = 4 - y] \end{aligned}$$

Thus  $f$  may have maxima or minima at  $(-1, 5)$  and  $(5, -1)$ .

Now

$$r = 6x, s = 12, t = 6y.$$

Points	(-7,-7)	(3,3)	(-1,5)	(5,-1)
--------	---------	-------	--------	--------

$r =$	-42	18	-6	30
$s =$	12	12	12	12
$t =$	-42	18	30	-6
$rt - s^2$	+ve	+ve	-ve	-ve
Result	Maximum	Minimum	neither Maxima nor Minima	neither Maxima nor Minima

Hence  $f$  has a maximum at  $(-7, -7)$  and a minima at  $(3, 3)$ .

$\therefore$  Maximum value of  $f = (-7)^3 + (-7)^3 - 63(-7 - 7) + 12(-7)(-7) = 784$   
and

$$\text{minimum value of } f = (3)^3 + (3)^3 - 63 \times 6 + 12 \times 3 \times 3 = -216$$

**Q 6.** Find a point within a triangle such that the sum of the squares of its dists, from the three vertices is a minimum.

**Solution.** Let  $(x_r, y_r), r = 1, 2, 3$  be the vertices of the triangle  $ABC$  and  $P(x, y)$  be any point inside the triangle.

Let

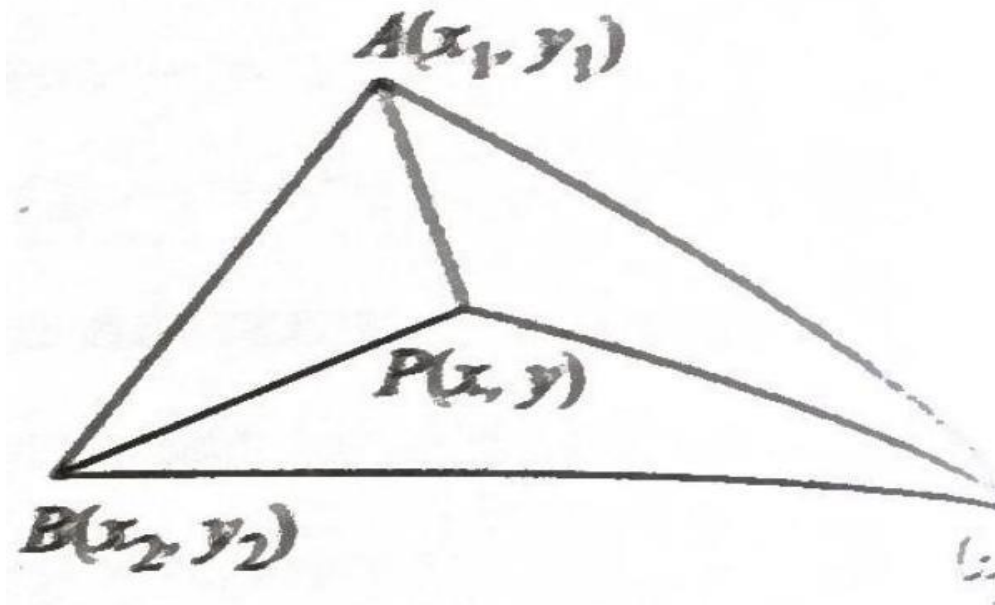
$$u = \sum_{r=1}^3 [(x - x_r)^2 + (y - y_r)^2]$$

For maxima or minima of  $u$ , we have

$$\frac{\partial u}{\partial x} = \sum 2(x - x_r) = 0$$

i.e.,

$$(x - x_1) + (x - x_2) + (x - x_3) = 0$$



or

$$x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly,

$$\frac{\partial u}{\partial y} = 0 \Rightarrow y = \frac{y_1 + y_2 + y_3}{3}$$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 6, s = \frac{\partial^2 u}{\partial y \partial x} = 0, t = \frac{\partial^2 u}{\partial y^2} = 6$$

So that

$$rt - s^2 = 36 = +ve$$

and  $r$  is tve. Hence  $u$  is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, y = \frac{y_1 + y_2 + y_3}{3}$$

Thus the required point is  $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ , which is the centroid of the triangle.

**Q 7.** Show that the distance of any point  $(x, y, z)$  on the plane  $2x + 3y - z = 12$  from the origin is given by  $l = \sqrt{x^2 + y^2 + (2x + 3y - 12)^2}$ .

Hence find the point on the plane that is nearest to the origin.

**Solution.** If  $l$  is the distance from  $(0,0,0)$  of any point  $(x, y, z)$ , then  $l = \sqrt{x^2 + y^2 + z^2}$ . If the point  $(x, y, z)$  lies on the plane  $2x + 3y - z = 12$ , then

$$l = \sqrt{x^2 + y^2 + (2x + 3y - 12)^2}$$

$[\because z = 2x + 3y - 12 \text{ from the equation of the plane}]$

$$l^2 = x^2 + y^2 + (2x + 3y - 12)^2$$

$$= 5x^2 + 10y^2 + 12xy - 48x - 72y + 144 = u, \text{ say}$$

Now  $l$  is maximum or minimum according as  $l^2$  i.e.,  $u$  is maximum or minimum. For a maximum or minimum of  $u$ , we have

$$\frac{\partial u}{\partial x} = 10x + 12y - 48 = 0 \text{ and } \frac{\partial u}{\partial y} = 20y + 12x - 72 = 0$$

Solving, we have  $x = 12/7$  and  $y = 18/7$ .

Also  $\therefore r = \frac{\partial^2 u}{\partial x^2} = 10, s = \frac{\partial^2 u}{\partial x \partial y} = 12$  and  $t = \frac{\partial^2 u}{\partial y^2} = 20$ .

$\therefore rt - s^2 = 10 \times 20 - 12^2 = +ve.$

Since  $rt - s^2 > 0$  and  $r > 0$ , therefore  $u$  is minimum and hence  $l$  is minimum, when  $x = 12/7$  and  $y = 18/7$ . Putting these values of  $x$  and  $y$  in the equation of the plane, we get  $z = 2(12/7) + 3(18/7) - 12 = -6/7$ . Hence the required point is  $(12/7, 18/7, -6/7)$ .

**Q 8.** Find the maximum, value of  $u$  where

**Solution.** Here  $u = x^2 + y^2 + z^2 + x - 2z - xy$ .

For maxima or minima of  $u$ , we have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0, \frac{\partial u}{\partial y} = -x + 2y = 0, \text{ and } \frac{\partial u}{\partial z} = 2z - 2 = 0.$$

Solving these equations, we get  $x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1$ .

At  $(-\frac{2}{3}, -\frac{1}{3}, 1)$ , we have

$$A = \frac{\partial^2 u}{\partial x^2} = 2, B = \frac{\partial^2 u}{\partial y^2} = 2, C = \frac{\partial^2 u}{\partial z^2} = 2$$

$$F = \frac{\partial^2 u}{\partial y \partial z} = 0, G = \frac{\partial^2 u}{\partial z \partial x} = 0, H = \frac{\partial^2 u}{\partial x \partial y} = -1$$

Now

$$A = 2 \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

and

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \times 3 = 6$$

Since all are positive, therefore,  $u$  is minimum at  $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ .



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**THANK YOU**